Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 3

Tuesday 11 May 2021 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

2221 – 7213

7 pages

Answer **both** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 24]

Juliet is a sociologist who wants to investigate if income affects happiness amongst doctors. This question asks you to review Juliet's methods and conclusions.

Juliet obtained a list of email addresses of doctors who work in her city. She contacted them and asked them to fill in an anonymous questionnaire. Participants were asked to state their annual income and to respond to a set of questions. The responses were used to determine a *happiness score* out of 100. Of the 415 doctors on the list, 11 replied.

(a) (i) Describe **one** way in which Juliet could improve the reliability of her investigation. [1]

(ii) Describe **one** criticism that can be made about the validity of Juliet's investigation. [1]

Juliet's results are summarized in the following table.

Response	Annual income ($)	Happiness score
A	65 000	60
B	63 000	52
C	40 000	31
D	125 000	81
E	100 000	48
F	245 000	61
G	48 000	42
H	39 000	40
I	85 000	57
J	92 000	53
K	123 456 789	56

(b) Juliet classifies response K as an outlier and removes it from the data. Suggest **one** possible justification for her decision to remove it. [1]

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(Question 1 continued)

(c) For the remaining ten responses in the table, Juliet calculates the mean happiness score to be 52.5.

(i) Calculate the mean **annual income** for these remaining responses. [2]

(ii) Determine the value of r, Pearson's product-moment correlation coefficient, for these remaining responses. [2]

Juliet decides to carry out a hypothesis test on the correlation coefficient to investigate whether increased annual income is associated with greater happiness.

(d) (i) State why the hypothesis test should be one-tailed. [1]

(ii) State the null and alternative hypotheses for this test. [2]

The critical value for this test, at the 5% significance level, is 0.549. Juliet assumes that the population is bivariate normal.

(iii) Determine whether there is significant evidence of a positive correlation between annual income and happiness. Justify your answer. [2]

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(Question 1 continued)

(e) Juliet wants to create a model to predict how changing annual income might affect happiness scores. To do this, she assumes that annual income in dollars, X, is the independent variable and the happiness score, Y, is the dependent variable.

She first considers a linear model of the form

$$Y = aX + b.$$

(i) Use Juliet's data to find the value of a and of b. [1]

(ii) Interpret, referring to income and happiness, what the value of a represents. [1]

Juliet then considers a quadratic model of the form

$$Y = cX^2 + dX + e.$$

(iii) Find the value of c, of d and of e. [1]

(iv) Find the coefficient of determination for each of the two models she considers. [2]

(v) Hence compare the two models. [1]

Juliet decides to use the coefficient of determination to choose between these two models.

(vi) Comment on the validity of her decision. [1]

After presenting the results of her investigation, a colleague questions whether Juliet's sample is representative of all doctors in the city.

A report states that the mean annual income of doctors in the city is $\$80\,000$. Juliet decides to carry out a test to determine whether her sample could realistically be taken from a population with a mean of $\$80\,000$.

(f) (i) State the name of the test which Juliet should use. [1]

(ii) State the null and alternative hypotheses for this test. [1]

(iii) Perform the test, using a 5% significance level, and state your conclusion in context. [3]

2. [Maximum mark: 31]

Alessia is an ecologist working for Mediterranean fishing authorities. She is interested in whether the mackerel population density is likely to fall below 5000 mackerel per km^3, as this is the minimum value required for sustainable fishing. She believes that the primary factor affecting the mackerel population is the interaction of mackerel with sharks, their main predator.

The population densities of mackerel (M thousands per km^3) and sharks (S per km^3) in the Mediterranean Sea are modelled by the coupled differential equations:

$$\frac{\text{d}M}{\text{d}t} = \alpha M - \beta MS$$

$$\frac{\text{d}S}{\text{d}t} = \gamma MS - \delta S$$

where t is measured in years, and α, β, γ and δ are parameters.

This model assumes that no other factors affect the mackerel or shark population densities.

The term αM models the population growth rate of the mackerel in the absence of sharks. The term βMS models the death rate of the mackerel due to being eaten by sharks.

(a) Suggest similar interpretations for the following terms.

(i) γMS [1]

(ii) δS [1]

(b) An equilibrium point is a set of values of M and S, such that $\frac{\text{d}M}{\text{d}t} = 0$ and $\frac{\text{d}S}{\text{d}t} = 0$.

Given that both species are present at the equilibrium point,

(i) show that, at the equilibrium point, the value of the mackerel population density is $\frac{\delta}{\gamma}$; [3]

(ii) find the value of the shark population density at the equilibrium point. [2]

(c) The equilibrium point found in part (b) gives the average values of M and S over time.

Use the model to predict how the following events would affect the average value of M. Justify your answers.

(i) Toxic sewage is added to the Mediterranean Sea. Alessia claims this reduces the shark population growth rate and hence the value of γ is halved. No other parameter changes. [2]

(ii) Global warming increases the temperature of the Mediterranean Sea. Alessia claims that this promotes the mackerel population growth rate and hence the value of α is doubled. No other parameter changes. [2]

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(Question 2 continued)

(d) To estimate the value of α, Alessia considers a situation where there are no sharks and the initial mackerel population density is M_0.

(i) Write down the differential equation for M that models this situation. [1]

(ii) Show that the expression for the mackerel population density after t years is $M = M_0 e^{\alpha t}$. [4]

(iii) Alessia estimates that the mackerel population density increases by a factor of three every two years. Show that $\alpha = 0.549$ to three significant figures. [3]

Based on additional observations, it is believed that

$$\alpha = 0.549,$$

$$\beta = 0.236,$$

$$\gamma = 0.244,$$

$$\delta = 1.39.$$

Alessia decides to use Euler's method to estimate future mackerel and shark population densities. The initial population densities are estimated to be $M_0 = 5.7$ and $S_0 = 2$.
She uses a step length of 0.1 years.

(e) (i) Write down expressions for M_{n+1} and S_{n+1} in terms of M_n and S_n. [3]

(ii) Use Euler's method to find an estimate for the mackerel population density after one year. [2]

(f) Alessia will use her model to estimate whether the mackerel population density is likely to fall below the minimum value required for sustainable fishing, 5000 per km^3, during the first nine years.

(i) Use Euler's method to sketch the trajectory of the phase portrait, for $4 \leq M \leq 7$ and $1.5 \leq S \leq 3$, over the first nine years. [3]

(ii) Using your phase portrait, or otherwise, determine whether the mackerel population density would be sufficient to support sustainable fishing during the first nine years. [2]

(iii) State **two** reasons why Alessia's conclusion, found in part (f)(ii), might not be valid. [2]

References: